

BETTI NUMBERS IN THREE DIMENSIONAL MINIMAL MODEL PROGRAM

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ABSTRACT. Let X be a smooth projective threefold. We prove that, among the process of minimal model program, the variance of the third Betti number can be bounded by some integer depends only on the Picard number of X .

1. INTRODUCTION

Minimal model program plays an important role in birational geometry. Given a smooth projective threefold, the minimal model program produces a finite sequence of birational maps, including divisorial contractions and flips. The final object is either a minimal model, that is, a projective threefold with at most terminal singularity and nef canonical divisor, or a Mori fiber space, which is a fibration with relatively ample canonical divisor. It is a natural and interesting question to compare the original variety and its minimal model. Thanks to the recent attempt of understanding three dimensional minimal model program, an explicit description of elementary birational maps between threefolds is known. With these works, it is thus possible to compare invariants between birational equivalent models of threefolds.

In this paper we compare the change of Betti numbers under the process of minimal model program. Betti numbers are important topological invariants of algebraic varieties and it may be used to bound some geometrical invariants. For example, in [CT], Cascini and Tasin use Betti numbers to bound $K_Y^3 - K_X^3$ where $Y \rightarrow X$ is a step of minimal model program begin with a smooth threefold X_0 . If $Y \rightarrow X$ is a divisorial contraction to point, then $K_Y^3 - K_X^3$ can be bounded by $2^{10}b_2(X_0)$. If $Y \rightarrow X$ is blow-up smooth curve, then $K_Y^3 - K_X^3$ can be bounded by some constant depends only on $b_3(Y)$ and the cubic form of Y .

In fact we are motivated by a lemma in [CT].

Lemma 1.1 ([CT], Lemma 2.17). *Let $Y \rightarrow X$ be an elementary divisorial contraction within \mathbb{Q} -factorial projective threefolds with terminal singularities. Then $b_i(Y) = b_i(X)$ if $i = 0, 1, 5, 6$, and $b_i(Y) = b_i(X) + 1$ if $i = 2, 4$.*

Under divisorial contractions all the Betti numbers vary regularly except for b_3 . A natural question is: how does b_3 change, and is there the same phenomenon for flips? In this article we answer this question.

Theorem 1.2. *Let X be a smooth threefold and $X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_m = X_{min}$ be a process of minimal model program. Then*

- (i) $b_i(X_j) = b_i(X)$ for $i = 0, 1, 5, 6$ and for all j .
- (ii) If $j > k$, then $b_i(X_j) \leq b_i(X_k)$ for $i = 2, 4$. Equality holds if and only if X_j and X_k are connected by flips.
- (iii) There exists an integer $\bar{\Phi}_{\rho(X)}$ depends only on the Picard number of X , such that $b_3(X_j) \leq \bar{\Phi}_{\rho(X)} + b_3(X)$ for all j .

We first deal with the divisorial contraction case. Since all other Betti numbers are known, to look at the variance of b_3 is equivalence to look at the variance of the topological Euler characteristic and, to find the change of topological Euler characteristic is equivalent to find the topological Euler characteristic of the exceptional divisor. Thanks for the classification of extremal divisorial contractions to points due to Hayakawa, Kawakita and Yamamoto, cf. [Hay1], [Hay2], [Hay3], [Hay4], [Hay5], [Kaw1], [Kaw2], [Kaw3] and [Yam], every extremal divisorial contraction to point can be viewed as a weighted blow-up of LCI locus in a cyclic quotient of \mathbb{A}^4 or \mathbb{A}^5 and hence the exceptional divisor will be a LCI locus in some weighted projective spaces. So the first step to solve our problem is to estimate the topological Euler characteristic of varieties in weighted projective spaces.

The main technical ingredient of our work is the following.

Theorem 1.3. *Fix three positive integers n , k and d .*

- (i) *There is an integer $N_{d,k}^n$ such that for any algebraic set $X_I \subset \mathbb{A}^n$ defined by an ideal $I = (f_1, \dots, f_k)$ with $\deg f_i \leq d$ for all i , we have $|\chi_{top}(X_I)| \leq N_{d,k}^n$.*
- (ii) *There is an integer $M_{d,k}^n$ such that for any zero locus $Y_I \subset \mathbb{P}(a_0, \dots, a_n)$ defined by an weighted homogeneous ideal $I = (f_1, \dots, f_k)$ with $wt(f_i) \leq d$ for all i , we have $|\chi_{top}(Y_I)| \leq M_{d,k}^n$, for arbitrary integer a_i , $i = 0, \dots, n$.*

With this theorem, one could estimate the change of b_3 after divisorial contractions to point. To go further, we use the factorization in [CH], which factorizes any extremal birational maps into composition of divisorial contractions to point, blow-up LCI curves, flops and the inverse of maps above. The Betti numbers won't change after flop, and the change of b_3 after blowing-up LCI curve can be easily computed. So the problem could be solved.

As an application of Theorem 1.2, we try to bound the intersection Betti numbers. Intersection homology was developed by Mark Goresky and Robert MacPherson in 1970's, which is defined on singular manifolds and satisfied some nice properties as original singular homology on smooth manifolds. One may expect that the difference of original Betti number and the intersection Betti number can be controlled by the singularity. In this paper we prove a weaker statement. We will denote by $IH^i(X, \mathbb{Q})$ the middle-perversity intersection cohomology group and let $Ib_i(X)$ be the dimension of $IH^i(X, \mathbb{Q})$.

Theorem 1.4. *Let X be a projective \mathbb{Q} -factorial terminal threefold over \mathbb{C} . Then there is an integer Θ_i depends only on the singularity of X and the Picard number $\rho(X)$, such that*

$$Ib_i(X) \leq b_i(X) + \Theta_i.$$

The idea is to compare the both two kinds of Betti numbers with a smooth model. On smooth model the two Betti numbers coincide. Thanks to the main theorem of [C], there is a smooth variety Y such that $Y \rightarrow X$ is a composition of extremal divisorial contractions, so Theorem 1.2 applies. As stated in [CT] the intersection Betti numbers always decrease under divisorial contractions, hence there is no difficulty to derive Theorem 1.4.

There are some question remained. In the prove of Theorem 1.4 the Picard number plays an essential role. However in the topological viewpoint it seems no reason that this term should appear. Can one find another bound which is only depends on the singularities? The other problem is that the converse of the inequality are much interesting. Can one prove that $b_i(X) \leq Ib_i(X) + \Theta'_i$ for some Θ'_i depends only on singularities? Since intersection Betti numbers plays the same role on singular varieties as original Betti numbers on smooth

varieties, one may regard the intersection Betti number as an elementary quantum on singular varieties. Then, use this quantum to bound the original Betti number seems more natural.

This article is structured as follows. In Section 2 we will quickly review some facts about terminal threefolds, involving the factorization of [CH], the discussion of invariants in terminal threefolds and the computation of Betti numbers for some easy cases. We will prove Theorem 1.3 in Section 3 and Theorem 1.2 in Section 4. The last section consists some examples and the proof of Theorem 1.4.

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2. PRELIMINARIES

2.1. Geometry of terminal threefolds. We define several invariants of terminal threefold, which is useful when studying threefold geometry.

Definition. Let X be a terminal threefold. A w -morphism is a extremal divisorial contraction which contract exceptional divisor to a point of index $r > 1$, such that the discrepancy of the exceptional divisor is $1/r$.

The depth of X , denoted by $\text{dep}(X)$, is the minimal length of sequence of w -morphisms $X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X$, such that X_n is Gorenstein. Note that by [Hay2] Theorem 1.2, for any terminal threefold X , $\text{dep}(X)$ exists and is finite.

Proposition 2.1 ([CH], Proposition 2.15). *If $f : Y \supset E \rightarrow X \ni P$ be (the germ of) a divisorial contraction to a point. Then $\text{dep}(Y) \geq \text{dep}(X) - 1$.*

Proposition 2.2 ([CH], Proposition 3.8). *Let $X \rightarrow W$ be a flipping contraction and $X \dashrightarrow X'$ be the flip, then $\text{dep}(X) > \text{dep}(X')$.*

Remark 2.3. Let X be a terminal threefold. Then $\text{dep}(X) = 0$ if and only if X is Gorenstein. In this case, by Corollary 0.1 of [B], there is no flipping contraction. Also, if $X \rightarrow W$ is a divisorial contraction to curve, then X is obtained by blowing up a LCI curve on W (cf. [Cut], Theorem 4).

Definition. Let (X, P) be a germ of terminal threefold. It is known (cf. [Mo], Proposition 1b.3) that the singular point P can be deformed into cyclic quotient points P_1, \dots, P_k . The number k is called the *axial weight* of (X, P) and will be denoted by $\text{aw}(P \in X)$. One can define $\Xi(P \in X) = \sum_{i=1}^k \text{index}(P_i)$. We will write $\text{aw}(X) = \sum_{P \in \text{Sing}(X)} \text{aw}(P \in X)$ and $\Xi(X) = \sum_{P \in \text{Sing}(X)} \Xi(P \in X)$. It is obvious that $\text{aw}(X) < \Xi(X)$.

Lemma 2.4 ([CZ], Lemma 3.2). *Let X be a terminal projective variety of dimension 3, then $\Xi(X) \leq 2\text{dep}(X)$.*

Proposition 2.5 ([CZ], Proposition 3.3). *Let X be a smooth projective threefold and assume that*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_k = Z$$

is a sequence of steps for the K_X -minimal model program of X . Then $\Xi(Z) \leq 2\rho(X)$.

Remark 2.6. In the proof of [CZ] Proposition 3.3, one can see that $\text{dep}(Z) \leq \rho(X)$. We will use this result later.

The next important result is the factorization in [CH].

Theorem 2.7 ([CH], Theorem 3.3). *Let $g : X \subset C \rightarrow W \ni P$ be an extremal neighborhood which is isolated (resp. divisorial). If X is not Gorenstein, then we have a diagram*

$$\begin{array}{ccc}
 Y & \dashrightarrow & Y' \\
 \downarrow f & & \downarrow f' \\
 X & & X' \\
 & \searrow g & \swarrow g' \\
 & W &
 \end{array}$$

where $Y \dashrightarrow Y'$ consists of flips and flops over W , f is a w -morphism, f' is a divisorial contraction (resp. a divisorial contraction to a curve) and $g' : X' \rightarrow W$ is the flip of g (resp. g' is divisorial contraction to a point).

Remark 2.8. The diagram above satisfied more properties.

- (i) $\text{dep}(Y) = \text{dep}(X) - 1$. This is by the construction of Y in [CH].
- (ii) Assume that $Y \dashrightarrow Y'$ is decomposed into $Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \dots \dashrightarrow Y_l = Y'$, then $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i > 0$. This is the step 4 in the proof of Theorem 3.3 in [CH].

2.2. Topology of terminal threefolds. We will compute the change of Betti numbers under threefold birational maps in this subsection. All Betti numbers are known except for b_3 .

Lemma 2.9 ([CT], Lemma 2.17). *Let $Y \rightarrow X$ be an elementary divisorial contraction within \mathbb{Q} -factorial projective threefolds with terminal singularities. Then $b_i(Y) = b_i(X)$ if $i = 0, 1, 5, 6$, and $b_i(Y) = b_i(X) + 1$ if $i = 2, 4$.*

Corollary 2.10. *If $X \rightarrow W$ is extremal divisorial contraction, then*

$$b_3(W) - b_3(X) = \chi_{\text{top}}(X) - \chi_{\text{top}}(W) - 2.$$

Proposition 2.11. *Let X be a smooth three-fold and $X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_m = X_{\min}$ is the process of minimal model program. Then b_0, b_1, b_5 and b_6 are constant among the sequence $\{X_i\}$ and both b_2 and b_4 are decreasing. Moreover, b_2 and b_4 are strictly decrease by one if $X_i \rightarrow X_{i+1}$ is a divisorial contraction, and remain unchange if $X_i \dashrightarrow X_{i+1}$ is a flip.*

Proof. Proposition 2.9 asserts the divisorial contraction case. Assume that $X_i \dashrightarrow X_{i+1}$ is a flip. We will apply Theorem 2.7 and induction on $\text{dep}(X_i)$. One has the diagram

$$\begin{array}{ccc}
 Y & \dashrightarrow & Y' \\
 \downarrow f & & \downarrow f' \\
 X_i & & X' \\
 & \searrow g & \swarrow g' \\
 & W &
 \end{array}$$

Note that by Remark 2.8 we have $\text{dep}(Y) = \text{dep}(X) - 1$. One can write

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \dots \dashrightarrow Y_l = Y'$$

and $Y_j \dashrightarrow Y_{j+1}$ is a flip or flop for all j , hence $\text{dep}(Y_j) \geq \text{dep}(Y_{j+1})$ by Proposition 2.2. By induction hypothesis and the fact that Betti numbers are invariant after flop, we have

$$b_i(Y) = b_i(Y') \text{ for } i \neq 3.$$

Hence

$$b_i(X) = b_i(Y) = b_i(Y') = b_i(X') \text{ for } i = 0, 1, 5, 6$$

and

$$b_i(X) = b_i(Y) - 1 = b_i(Y') - 1 = b_i(X') \text{ for } i = 2, 4.$$

□

3. THE ESTIMATE ON TOPOLOGY

The purpose of this section is to prove Theorem 1.3. We shall prove that there is an integer $N_{d,k}^n$ such that any algebraic set in \mathbb{A}^n defined by k polynomials of degree $\leq d$ has topological Euler characteristic bounded by $N_{d,k}^n$. Similarly, there is an integer $M_{d,k}^n$ such that an algebraic set in a weighted projective space of dimension m which is defined by k weighted homogeneous polynomials of weight $\leq d$ has topological Euler characteristic bounded by this integer. To prove the existence of such kind of integers, the basic idea is to reduce the question into lower dimensional cases. We will prove:

Proposition 3.1. *Assume that $N_{e,l}^{n-1}$ exists for all $e, l \in \mathbb{N}$, then $N_{d,k}^n$ exists.*

Proposition 3.2. *If $M_{d,k}^m$ exists for all $m < n$ and $N_{d,l}^n$ exists for all $l \geq k$, then $M_{d,k}^n$ exists.*

3.1. The existence of N -constant. In this subsection we prove Proposition 3.1. Given $X_I \subset \mathbb{A}^n$, where $I = (f_1, \dots, f_k)$ satisfying $\deg f_i \leq d$. Consider the natural map

$$K[x_1, \dots, x_{n-1}] \hookrightarrow K[x_1, \dots, x_n] \rightarrow K[x_1, \dots, x_n]/I,$$

here K is the ground field. This gives a morphism ϕ from X_I to $\{x_n = 0\} \cong \mathbb{A}^{n-1}$. Fix $p = (a_1, \dots, a_{n-1}) \in \mathbb{A}^{n-1}$, then

$$\phi^{-1}(p) = \{(a_1, \dots, a_{n-1}, x_n) \in \mathbb{A}^n \mid f_1(a_1, \dots, a_{n-1}, x_n) = \dots = f_k(a_1, \dots, a_{n-1}, x_n) = 0\}.$$

Thus $\phi^{-1}(p)$ can be studied via the equations f_1, \dots, f_k . Now assume that the topology of the image is known, then since the fibres can be studied, the topology of the original space X_I could be computed. This is the reason that one can reduce the problem to the lower dimensional case.

For the induction reason, we will prove a stronger statement.

Proposition 3.3. *Assume $N_{c,m}^{n-1}$ exists for all integers c and m . Let Z be an algebraic subset in \mathbb{A}^{n-1} which is defined by an ideal $J = (g_1, \dots, g_l)$ and assuming $\deg g_j \leq e$ for some constant e . Then there is a fixed integer $L_{d,k,e,l}^n$ such that $|\chi_{\text{top}}(\phi^{-1}Z)| \leq L_{d,k,e,l}^n$.*

We divide this subsection into four parts. In the first part we study the common roots of a collection of polynomials, which is the main tool we will use to study the fibre of the projection ϕ . After the tool is developed, we could get much information between the points in H and its fibre in \mathbb{A}^n , provided the degree of f_1, \dots, f_k do not be too small. This is the second part of this subsection. In the third part we deal with the case when the degree of f_i is too small for some i so that above technique does not work. Finally in the last part we run a complicated induction and prove Proposition 3.3.

3.1.1. *The generalized resultant.* We generalize the idea of the *resultant* in classical algebra to describe the condition that a collection of polynomials has a common zero.

Let $g_1, \dots, g_k \in K[x]$ be one variable polynomials with $\deg g_i = d_i > 0$. One write $g_i = \sum_j a_{i,j} x^j$ and we will denote

$$A_{g_1, \dots, g_k}^i = \begin{pmatrix} a_{i,d_i} & & & \\ a_{i,d_i-1} & a_{i,d_i} & & \\ \vdots & \vdots & \ddots & a_{i,d_i} \\ a_{i,0} & \vdots & \ddots & \vdots \\ & a_{i,0} & & \vdots \\ & & & a_{i,0} \end{pmatrix}$$

which is a $(d_i + d_k) \times d_k$ matrix satisfying

$$(A_{g_1, \dots, g_k}^i)_{pq} = \begin{cases} a_{i,d_i-q+p} & 0 \leq p - q \leq d_i \\ 0 & \text{otherwise} \end{cases}.$$

Also define

$$B_{g_1, \dots, g_k}^i = \begin{pmatrix} a_{k,d_k} & & & \\ a_{k,d_k-1} & a_{k,d_k} & & \\ \vdots & \vdots & \ddots & a_{k,d_k} \\ a_{k,0} & \vdots & \ddots & \vdots \\ & a_{k,0} & & \vdots \\ & & & a_{k,0} \end{pmatrix}$$

be a $(d_i + d_k) \times d_i$ matrix such that

$$(B_{g_1, \dots, g_k}^i)_{pq} = \begin{cases} a_{k,d_k-q+p} & 0 \leq p - q \leq d_k \\ 0 & \text{otherwise} \end{cases}.$$

Consider

$$T_{g_1, \dots, g_k} = \begin{pmatrix} A_{g_1, \dots, g_k}^1 & B_{g_1, \dots, g_k}^1 & 0 & \cdots & 0 \\ A_{g_1, \dots, g_k}^2 & 0 & B_{g_1, \dots, g_k}^2 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ A_{g_1, \dots, g_k}^{k-1} & 0 & \cdots & 0 & B_{g_1, \dots, g_k}^{k-1} \end{pmatrix},$$

which is a $(d_1 + \dots + d_{k-1} + (k-1)d_k) \times (d_1 + \dots + d_k)$ matrix.

Lemma 3.4. *The polynomials g_1, \dots, g_k have common zeros if and only if the matrix T_{g_1, \dots, g_k} is not full rank. Moreover, the number of the common zeros is exactly the nullity of T_{g_1, \dots, g_k} , counted with multiplicity.*

Proof.

Claim. g_1, \dots, g_k has common zero if and only if there is polynomials h_1, \dots, h_k such that $\deg h_i < \deg g_i$ and $h_i g_k = h_k g_i$ for all $i < k$.

Indeed, if the polynomials has common zeros, then they have a common factor in the polynomial ring $K[x]$. So we may write $g_i = b h_i$, where $b = \gcd(g_1, \dots, g_k)$ and then $\deg h_i < \deg g_i$ and $h_i g_k = h_k g_i$. Conversely, assume $h_i g_k = h_k g_i$ for some h_1, \dots, h_k with $\deg h_i < \deg g_i$. If g_k and h_k has no common root, then every root of g_k is a root of g_i for all i thanks

to the relation $h_i g_k = h_k g_i$. Otherwise let $l = \gcd(g_k, h_k)$ and define $\bar{g}_k = g_k/l$, $\bar{h}_k = h_k/l$. Then $\deg \bar{g}_k > 0$. We still have the relation $h_i \bar{g}_k = \bar{h}_k g_i$ and $\gcd(\bar{g}_k, \bar{h}_k) = 1$. As the previous discussion the root of \bar{g}_k will be a root of g_i for all i .

Thus to prove the lemma, one only need to find h_i satisfied the condition above. Let

$$v = (r_{k,d_k-1}, \dots, r_{k,0}, -r_{1,d_1-1}, \dots, -r_{1,0}, -r_{2,d_2-1}, \dots, -r_{k-1,d_{k-1}-1}, \dots, -r_{k-1,0})^t$$

be a column vector in $K^{d_1+\dots+d_k}$, and let $h_i = \sum_j r_{i,j} x^j$, then one can check that the condition $h_i g_k = h_k g_i$ is exactly the linear condition $T_{g_1, \dots, g_k} v = 0$. Hence g_1, \dots, g_k has common zeros if and only if T_{g_1, \dots, g_k} is not full rank.

Now notice that if $b = \gcd(g_1, \dots, g_k)$ and let $\alpha_i = g_i/b$, then the number of common zeros of g_1, \dots, g_k is exactly $\deg b$. For $1 \leq j \leq \deg b$, Let v_j be the vector in $K^{d_1+\dots+d_k}$ corresponds to the collection of polynomials $\{x^{j-1} \alpha_i\}_{i=1}^k$, then v_j is lying on the null space of M and $v_1, \dots, v_{\deg b}$ are linearly independent.

Conversely assume $T_{g_1, \dots, g_k} w = 0$ for some $w \in K^{d_1+\dots+d_k}$, then w corresponds to a collection of polynomials h_1, \dots, h_k satisfying $h_i g_k = h_k g_i$ and $\deg h_i < \deg g_i$. We claim that α_i divides h_i for all i .

Let $c_i = \gcd(g_i, g_k)$, $g_i = c_i \beta_i$ and $g_k = c_i \gamma_i$. The relation $h_i g_k = h_k g_i$ yields $h_i \gamma_i = h_k \beta_i$. Since $\gcd(\beta_i, \gamma_i) = 1$ we have γ_i divides h_k for all i , hence $\text{l.c.m.}(\gamma_1, \dots, \gamma_{k-1})$ divides h_k . On the other hand we have the relation $g_k = b \alpha_k = c_i \gamma_i$. Note that $b = \gcd(c_1, \dots, c_{k-1})$, hence γ_i divide α_k for all i and so $\text{l.c.m.}(\gamma_1, \dots, \gamma_{k-1})$ divides α_k . If $\alpha_k \neq \text{l.c.m.}(\gamma_1, \dots, \gamma_{k-1})$ then $\alpha_k / \text{l.c.m.}(\gamma_1, \dots, \gamma_{k-1})$ will divide c_i for all i , contradict to $b = \gcd(c_1, \dots, c_{k-1})$. Thus $\alpha_k = \text{l.c.m.}(\gamma_1, \dots, \gamma_{k-1})$ divides h_k . Finally the relation $h_i g_k = h_k g_i$ gives that $h_i \alpha_k = h_k \alpha_i$. Since α_k divide h_k , we have α_i divide h_i for all i .

Now $\deg h_i < \deg g_i = \deg b + \deg \alpha_i$, hence $h_i = h' \alpha_i$ for some polynomial h' and $\deg h' < \deg b$. Thus w is lying on the subspace generated by $v_1, \dots, v_{\deg b}$ and then $\text{null}(T_{g_1, \dots, g_k}) = \deg b$ and the last part of the lemma is proved. \square

Lemma 3.5. Assume $\deg g_i > 1$ for all i . Let

$$s_0 = \text{null}(T_{g_1, \dots, g_k}); \quad s_1 = \text{null}(T_{g_1, \dots, g_k, g'_1, \dots, g'_k}),$$

here g'_i denotes the formal derivative of polynomials. Then the number of distinct common roots of g_1, \dots, g_k is exactly $s_0 - s_1$.

Proof. Let $b = \gcd(g_1, \dots, g_k)$. We will show that $\text{g.c.d.}(b, b') = \gcd(g_1, \dots, g_k, g'_1, \dots, g'_k)$. Indeed, if we write $g_i = b h_i$, then $g'_i = b' h_i + b h'_i$, hence $\text{g.c.d.}(b, b')$ divides g_i and g'_i for all i and then $\gcd(b, b')$ divides $\gcd(g_1, \dots, g_k, g'_1, \dots, g'_k)$. Conversely, if p is a polynomial divides g_i and g'_i for all i , then p will divide $\gcd(g_1, \dots, g_k) = b$. The condition p divides g'_i implies p divides $b' h_i$ for all i . However, $\gcd(h_1, \dots, h_k) = 1$. Thus p divides b' and hence p divides $\gcd(b, b')$. That is, $\gcd(g_1, \dots, g_k, g'_1, \dots, g'_k)$ divides $\gcd(b, b')$.

Now write $b = (x - a_1)^{r_1} \dots (x - a_m)^{r_m}$, then the number of distinct common roots of g_1, \dots, g_k is m . On the other hand,

$$b' = ((x - a_1)^{r_1-1} \dots (x - a_m)^{r_m-1}) \left(\sum_i r_i (x - a_1) \dots (x - a_{i-1}) (x - a_{i+1}) \dots (x - a_m) \right).$$

Hence $\gcd(b, b') = (x - a_1)^{r_1-1} \dots (x - a_m)^{r_m-1}$. By Lemma 3.4, $s_0 = \deg b = r_1 + \dots + r_m$ and $s_1 = \deg(\gcd(b, b')) = (r_1 - 1) + \dots + (r_m - 1) = r_1 + \dots + r_m - m$. A conclusion is that $s_0 - s_1 = m$, as we want. \square

3.1.2. *The geometry of the projection map.* In this part we study the fibre of $\phi : X_I \rightarrow \mathbb{A}^{n-1}$. We will view f_i as a polynomial in x_n and we will denote $f'_i = \frac{\partial}{\partial x_n} f_i$. Let

$$T^0 = \begin{cases} T_{f_1, f'_1} & \text{if } k = 1, \\ T_{f_1, \dots, f_k} & \text{if } k > 1. \end{cases} \quad T^1 = \begin{cases} T_{f_1, f'_1, f''_1} & \text{if } k = 1, \\ T_{f_1, \dots, f_k, f'_1, \dots, f'_k} & \text{if } k > 1. \end{cases}$$

provided that all the polynomials are non-constant. Note that T^0 and T^1 are matrices with all entries being a polynomial in $K[x_1, \dots, x_{n-1}]$.

Convention. For $j = 0, 1$, we say the *condition* (A^j) are satisfied if T^j is defined. That is, $\deg f_i > j$ (resp. $j + 1$) for all i if $k > 1$ (resp. $k = 1$).

When (A^j) is satisfied, one could study the fiber of ϕ via the nullity of T^j . There are three possibility of the fiber of ϕ : empty, finite points or a \mathbb{A}^1 . The fiber is a \mathbb{A}^1 at a point $P \in \mathbb{A}^{n-1}$ if and only if all f_i vanishes at P , which is easy to detect. The main question is to find the locus on \mathbb{A}^{n-1} such that the pre-image of ϕ is finite, and on such locus one should find the number of points in the fiber.

Assume (A^0) one could solve the first question (cf. Lemma 3.6, Lemma 3.7). If (A^1) holds and assuming more conditions one could count the cardinality of the fiber (cf. Lemma 3.8).

Lemma 3.6. *Assume (A^0) . Fix $p \in \mathbb{A}^{n-1}$ and assume that $T^0(p)$ is full rank. Then*

$$|\phi^{-1}(p)| = \begin{cases} \deg f_1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

Proof. Assume $k = 1$. If the leading coefficient vanishes over p , then the first row of T^0 is always zero. Since T^0 is a square matrix, this implies T^0 is not full rank. Hence we may assume the leading coefficient do not vanishing at p , so both f_1 and f'_1 are non-constant. Using Lemma 3.4, we see that $T^0(p)$ is full rank implies f_1 and f'_1 consist no common zero. Hence f_1 consists no multiple roots over p , so $|\phi^{-1}(p)| = \deg f_1$.

Now assume $k > 1$. First assume f_i is constant over p for some i . Then if f_i is identically zero, T^0 can not be full rank. On the other hand, if f_i is a non-zero constant, then $\phi^{-1}(p)$ is always empty so the conclusion is always true. Finally assume f_i is non-constant for all i , then for any $p \in H$, $\phi^{-1}(p)$ is non-empty only if f_1, \dots, f_k admit common zeros. By Lemma 3.4, this implies the matrix T^0 is not full rank. \square

Lemma 3.7. *Assume (A^0) . Given $p \in \mathbb{A}^{n-1}$ and assume that $T^0(p)$ is not full rank. Assume further that the leading coefficient of f_i do not vanish at p for all i . Then if $k > 1$, we have that p is contained in the image of ϕ . For $k = 1$, one can say that ϕ is a finite morphism near p and p is lying on the ramification locus.*

Proof. First assume $k > 1$. The hypothesis implies that f_1, \dots, f_k is non-constant polynomial in x_n over p . By Lemma 3.4, T^0 is not full rank at p if and only if f_1, \dots, f_k admits a common zero, say $\xi \in K$. If we write $p = (a_1, \dots, a_{n-1})$, then the point $(a_1, \dots, a_{n-1}, \xi)$ is lying on X_I and is mapped to p by ϕ . Hence p is contained in the image of ϕ .

For the $k = 1$ case, note that T^0 is defined implies $\deg f_1 > 1$. By assumption, the leading coefficient of f_1 do not vanish at p , hence it do not vanish on a neighborhood U of p . We see that for any point $q \in U$ we have f_1 is a polynomial of positive degree in x_n over q , so the pre-image of ϕ consists only finitely many points and so ϕ is a finite morphism on U . Now the condition that $T^0(p)$ is not full rank implies f_1 consists multiple root over p , hence p is lying in the ramification locus of ϕ . \square

Now let $Z \subset \mathbb{A}^{n-1}$ be a subset contained in the image of ϕ . For $p \in Z$ we will denote $r(p) = |\phi^{-1}(p)|$ and $r(Z) = \max_{p \in Z} \{r(p)\}$. Also define $s_0(p) = \text{null}(T^0(p))$ and $s_1(p) = \text{null}(T^1(p))$. What we want to do is to find the locus which consists of the points $p \in Z$ such that $r(p) \neq r(Z)$. Such point could be determined using the number s_0 and s_1 , under suitable conditions.

Lemma 3.8. *Fix $Z \subset \mathbb{A}^{n-1}$ be any subset. Assume that the leading coefficient of f_i do not vanish over Z for all i . When $k = 1$ (resp. $k > 1$) assume (A^0) (resp. (A^1)). Then for any $p \in Z$ we have*

- (i) Assume $k = 1$, then $r(p) = \deg f_1(p) - s_0(p)$.
- (ii) Assume $k > 1$, then $r(p) = s_0(p) - s_1(p)$.

Proof. First assume $k > 1$. By Lemma 3.5 we have $r(p) = s_0(p) - s_1(p)$ for all $p \in Z$. Now assume $k = 1$. The assumption that T^0 exists and the leading coefficient of f_1 do not vanish implies that ϕ is a finite morphism over Z . For any p in Z the number $r(p)$ is the number of distinct roots of f_1 over p . Assume $f_1(p) = (x_n - a_1)^{r_1} \dots (x_n - a_m)^{r_m} (x_n - b_1) \dots (x_n - b_l)$ with $r_i > 1$. We have $r(p) = m + l$, $\deg f_1(p) = r_1 + \dots + r_m + l$, $s_0(p) = (r_1 - 1) + \dots + (r_m - 1) = r_1 + \dots + r_m - m$ by Lemma 3.4, hence $r(p) = \deg f_1 - s_0(p)$. \square

Corollary 3.9. *Fix $Z \subset \mathbb{A}^{n-1}$. Assume that the leading coefficient of f_i do not vanish over Z for all i and one of the following condition holds:*

- (i) $k = 1$ and (A^0) holds.
- (ii) $k > 1$, (A^1) holds and s_0 is constant over Z .

Then ϕ is a finite morphism over Z . When $k = 1$ (resp. $k > 1$) the ramification locus of ϕ is exactly the locus where the function s_0 (resp. s_1) do not reach its minimum.

3.1.3. The small degree cases. In this section we deal with the cases that the $\deg f_i$ is too small so that (A^0) or (A^1) dose not hold.

Lemma 3.10. *Under the assumption and notation in Proposition 3.3, if $k = 1$ and (A^0) dose not hold over Z , then the conclusion of Proposition 3.3 is true.*

Proof. The assumption says that $\deg f_1 < 2$ over Z . If $\deg f_1 = 0$, then $f_1 \in K[x_1, \dots, x_{n-1}]$ is independent of x_n . Let Z' be the zero locus of the ideal $J + (f_1)$, then $|\chi_{\text{top}}(Z')| \leq N_{\max\{e,d\},l+1}^{n-1}$. One see that outside Z' , the pre-image of ϕ is empty, and $\phi^{-1}Z' \cong Z' \times \mathbb{A}^1$. Hence $|\chi_{\text{top}}(\phi^{-1}Z)| = |\chi_{\text{top}}(\phi^{-1}Z')| = |\chi_{\text{top}}(Z')| \leq N_{\max\{e,d\},l+1}^{n-1}$.

On the other hand, assume $\deg f_1 = 1$. Write $f_1 = a_1 x_n + a_0$. Let Z_0 be the zero locus defined by $J + (a_1)$ and $Z_1 = Z - Z_0$. Then $\chi_{\text{top}}(\phi^{-1}Z_0)$ can be computed in the previous case since we can replace f_1 by a_0 and replace Z by Z_0 . On the other hand, since f_1 is a degree one polynomial over any points in Z_1 , we have $\phi^{-1}Z_1 \cong Z_1$. Now $|\chi_{\text{top}}(\phi^{-1}Z_1)| = |\chi_{\text{top}}(Z_1)| = |\chi_{\text{top}}(Z) - \chi_{\text{top}}(Z_0)| \leq N_{e,l}^{n-1} + N_{\max\{e,d\},l+1}^{n-1}$ can be compute. Thus the lemma is proved. \square

The other case is that (A^0) holds but (A^1) dose not hold. This happened when $k = 1$ and $\deg f_1 = 2$ or $k > 1$ and $\deg f_i = 1$ for some i .

Lemma 3.11. *Let $Z \subset H$ and assume the following.*

- (i) (A^0) holds but (A^1) dose not hold.
- (ii) $T^0(p)$ is not full rank for all $p \in Z$.

(iii) The leading coefficient of f_i do not vanishing for all i for any point $p \in Z$.

Then ϕ is one-to-one over Z . In particular, $\chi_{top}(\phi^{-1}Z) = \chi_{top}(Z)$.

Proof. First assume $k > 1$. By Lemma 3.7 the assumption yields that Z is contained in the image of ϕ . On the other hand, T^0 is defined but T^1 is not defined implies $\deg f_i = 1$ for some i , hence ϕ is one-to-one over Z .

Now assume $k = 1$. Since T^0 is defined but T^1 is not defined, we have $\deg f_1 = 2$, hence ϕ is two-to-one over some open neighborhood of Z . However, Lemma 3.7 implies that Z is lying on the ramification locus, hence ϕ is one-to-one over Z . \square

3.1.4. *The main proofs.* We will need the following lemma.

Lemma 3.12. *If $S = S_1 \cup S_2 \cup \dots \cup S_k$ for some algebraic set S_i . For any $I \subset \{1, \dots, k\}$, we denote $S_I = \bigcap_{i \in I} S_i$. Assume that $|\chi_{top}(S_I)| \leq M$ for some integer M and for all $I \subset \{1, \dots, k\}$. Then $|\chi_{top}(S)| \leq (2^k - 1)M$.*

Proof. We prove by induction on k . When $k = 2$ we have

$$\chi_{top}(S) = \chi_{top}(S_1 - S_{12}) + \chi_{top}(S_2 - S_{12}) + \chi_{top}(S_{12}) = \chi_{top}(S_1) + \chi_{top}(S_2) - \chi_{top}(S_{12}),$$

so $|\chi_{top}(S)| \leq 3M = (2^2 - 1)M$. In general let $S' = S_1 \cup \dots \cup S_{k-1}$, then $S' \cap S_k = (S_1 \cap S_k) \cup \dots \cup (S_{k-1} \cap S_k)$, hence $|\chi_{top}(S' \cap S_k)| \leq (2^{k-1} - 1)M$ by induction hypothesis. We also have $|\chi_{top}(S')| \leq (2^{k-1} - 1)M$. Thus

$$\begin{aligned} |\chi_{top}(S)| &= |\chi_{top}(S' - (S' \cap S_k)) + \chi_{top}(S_k - (S' \cap S_k)) + \chi_{top}(S' \cap S_k)| \\ &\leq |\chi_{top}(S')| + |\chi_{top}(S_k)| + |\chi_{top}(S' \cap S_k)| \\ &\leq (2(2^{k-1} - 1) + 1)M = (2^k - 1)M. \end{aligned}$$

\square

Proof of Proposition 3.3. We will divide Z into many pieces, and treat each piece separately. In each piece, either the topology of the pre-image can be easily computed, or after cut out some closed subset the pre-image can be computed, and there is some quantum which strictly decrease after restrict to the subset above. In the latter case we can use induction on the special quantum and finally the problem could be solved. We will treat the following cases.

Case(I) (A^0) holds.

Let

$$Z' = \{p \in Z \mid T^0(p) \text{ is not full rank} \}$$

and $Z'' = Z - Z'$. By Lemma 3.6,

$$\chi_{top}(\phi^{-1}Z'') = \begin{cases} (\deg f_1)\chi_{top}(Z'') & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

We further divide Z' into

$$Z_- = \{p \in Z' \mid \text{The leading coefficient of } f_i \text{ vanish over } p \text{ for some } i\}$$

and $Z_+ = Z' - Z_-$. To compute $\chi_{top}(\phi^{-1}Z_-)$, let a_i be the leading coefficient of f_i . For $S \subset \{1, \dots, k\}$, let W_S be the zero locus defined by $J + (a_{i_0} \dots a_{i_p})$ if $S = \{i_0, \dots, i_p\}$ and J is the defining ideal of Z . Then W_S is the locus in Z such that the leading coefficient of f_i vanish for all $i \in S$. Hence $Z_- = \bigcup_{1 \leq i \leq k} W_i$ and $W_S = \bigcap_{i \in S} W_i$.

Furthermore, one may induction on the number $\deg f_1 + \dots + \deg f_k$ so that we may assume $\chi_{top}(\phi^{-1}W_S)$ can be computed. By Lemma 3.12, $\chi_{top}(\phi^{-1}Z_-)$ can be bounded.

Now one has to compute $\chi_{top}(\phi^{-1}Z_+)$. If (A^1) is not true, then Lemma 3.11 implies that $\chi_{top}(\phi^{-1}Z_+) = \chi_{top}(Z_+)$. Assume (A^1) is true. We divide Z_+ into

$$Z_0 = \{p \in Z_+ \mid s_0(p) \text{ reach its minimum in } Z_+\}$$

and $Z'_0 = Z_+ - Z_0$. One may replace Z by Z'_0 and induction on $\min_{p \in Z} \{s_0(p)\}$. This number is increasing and always less or equal than $\deg f_i$ for all i , so after finite step, Z'_0 would be empty.

If $k > 1$ we further divide Z_0 into

$$Z_1 = \{p \in Z_0 \mid s_1(p) \text{ reach its minimum in } Z_0\}$$

and $Z'_1 = Z_0 - Z_1$. By Corollary 3.9, when $k = 1$ (resp. $k > 1$) ϕ is unramified over Z_0 (resp. Z_1). Hence

$$|\chi_{top}(\phi^{-1}Z_i)| = r(Z_i)|\chi_{top}(Z_i)| \leq d|\chi_{top}(Z_i)|,$$

with $i = 0$ (resp. $i = 1$) in $k = 1$ (resp. $k > 1$) case.

When $k > 1$ we have $r(Z'_1) < r(Z_0)$. We will replace Z by Z'_1 and induction on the number r . When $r(Z_0) = 1$ Z'_1 is always empty, so the induction works.

Case(II) (A^0) does not hold. If $k = 1$, this case can be solved by Lemma 3.10. Now assume $k > 1$. In this case $\deg f_i = 0$ for some i . If f_i is a non-zero constant, then $\phi^{-1}Z$ is empty, so there is nothing to prove. If f_i is identically zero, we can drop out f_i from the generator of I , and goes to the case with smaller k . By induction on k , this situation is solved.

We have to show that Z' , Z_- , Z'_1 and Z'_0 can be defined by algebraic equations, and the total number and the degree of those equations can be bounded by some integer depends on d and k , so the induction could work.

To see this, let c_i and r_i be the number of columns and rows of T^i , respectively, for $i = 0, 1$. Then $c_0 \leq dk$, $r_0 \leq 2(k-1)d$, and $c_1 \leq 2c_0$, $r_1 \leq 2r_0$. Let R be the ideal containing all maximal minors of T^0 , then R can be generated by $C_{c_0}^{r_0}$ many generators and each generator is a degree at most dr_0 polynomial. One can see that Z' is generated by $J + R$. Since $Z_- = \bigcup_{1 \leq i \leq k} W_i$ and the defining ideal of W_i are bounded, the defining ideal of Z_- is bounded.

Now let $t_i = \max_{p \in Z} rk(T^i(p))$. $p \in Z$ satisfied $s_i(p)$ do not reach minimum if and only if $s_i(p) + t_i > r_i$. Hence one only need to find those points in Z such that the rank of T^i at that point is less than t_i , or equivalently, all $t_i \times t_i$ minors of T^i vanishes. Let Q_i be the ideal containing all $t_i \times t_i$ minors of T^i , then Q_i is generated by at most $r_i c_i$ many elements and each element is a degree at most $dt_i \leq dr_i$ polynomial in $K[x_1, \dots, x_{n-1}]$. One can easily see that Z'_i is defined by $J + Q_i$ for $i = 0, 1$.

The other task is to compute $\chi_{top}(Z'')$, $\chi_{top}(Z_+)$ and $\chi_{top}(Z_i)$ for $i = 0, 1$. Since Z' is generated by $J + R$,

$$|\chi_{top}(Z')| \leq N_{e+dr_0, l+C_{c_0}^{r_0}}^{n-1}.$$

We have $\chi_{top}(Z'') = \chi_{top}(Z) - \chi_{top}(Z')$. Thus

$$|\chi_{top}(Z'')| \leq N_{e,l}^{n-1} + N_{e+dr_0, l+C_{c_0}^{r_0}}^{n-1}.$$

Now consider $|\chi_{top}(W_S)| \leq N_{e+d|S|,l+1}^{n-1} \leq N_{e+d^k,l+1}^{n-1}$ for all $S \subset \{1, \dots, k\}$, hence

$$|\chi_{top}(Z_-)| \leq (2^k - 1)N_{e+d^k,l+1}^{n-1}$$

by Lemma 3.12. A conclusion is that $\chi_{top}(Z_+) = \chi_{top}(Z') - \chi_{top}(Z_-)$ can be bounded.

Finally we try to bound $\chi_{top}(Z_i)$. As the argument above Z'_i is defined by the ideal $J + Q_i$ for $i = 0$ and 1, hence $|\chi_{top}(Z'_i)| \leq N_{e+dr_i,l+r_i c_r}^{n-1}$ can be bounded. Thus

$$\chi_{top}(Z_0) = \chi_{top}(Z_+) - \chi_{top}(Z'_0) \text{ and } \chi_{top}(Z_1) = \chi_{top}(Z_0) - \chi_{top}(Z'_1)$$

can be bounded. \square

Proof of Proposition 3.1. One can take $N_{d,k}^n = L_{d,k,0,1}^n$ by considering J in Proposition 3.3 to be the zero ideal. \square

3.2. The existence of M -constant.

Proof of Proposition 3.2. Given $Y = Y_I \subset \mathbb{P}(a_0, \dots, a_n)$, we may assume Y dose not contained in $\{a_0 = 0\}$. Let $Y' = Y \cap \{a_0 = 1\}$ and $Y'' = Y - Y'$, then $Y'' = Y \cap \{a_0 = 0\}$ can be viewed as the zero locus of a weighted homogeneous ideal in $\mathbb{P}(a_1, \dots, a_n)$, so $|\chi_{top}(Y'')| \leq M_{d,k}^{n-1}$.

On the other hand, $Y' \subset \mathbb{A}^n / \frac{1}{a_0}(a_1, \dots, a_n)$. Let \bar{Y} be the pre-image of Y' under the natural map $\mathbb{A}^n \rightarrow \mathbb{A}^n / \frac{1}{a_0}(a_1, \dots, a_n)$, then \bar{Y} is defined by an ideal I generated by k elements, and we may assume the degree of each generator of I is less or equal than d . Hence $|\chi_{top}(\bar{Y})| \leq N_{d,k}^n$.

Now $\bar{Y} \rightarrow Y'$ is a branched covering. Let $\bar{R} \subset \bar{Y}$ be the branched locus, and $R \subset Y'$ be the image of \bar{R} . One have to compute $\chi_{top}(R)$ and $\chi_{top}(\bar{R})$. Note that the morphism $A^n \rightarrow A^n / \frac{1}{a_0}(a_1, \dots, a_n)$ ramified at $\{x_{i_1} = \dots = x_{i_l} = 0\}$ for some i_1, \dots, i_l . Let Ξ_1, \dots, Ξ_j be the irreducible component of the ramification locus on \mathbb{A}^n and let $\bar{S}_i = \Xi_i \cap \bar{R}$. One can see that $|\chi_{top}(\bar{S}_i)| \leq N_{d,k+l_i}^n$ if $\Xi_i = \{x_{i_1} = \dots = x_{i_{l_i}} = 0\}$ and $|\chi_{top}(\bar{S}_{i_1} \cap \dots \cap \bar{S}_{i_m})| \leq N_{d,k+l'}^n$ if $\Xi_{i_1} \cap \dots \cap \Xi_{i_m}$ is of codimension l' . Moreover, the number of irreducible components of ramification locus of $A^n \rightarrow A^n / \frac{1}{a_0}(a_1, \dots, a_n)$ is less than $\max_{2 \leq m \leq n} \{C_m^n\} < 2^n$, which is a number depends only on n . Hence by Lemma 3.12, there is an integer $A_{d,k}^n$ depends on n, d and k such that $|\chi_{top}(\bar{R})| \leq A_{d,k}^n$.

To find $\chi_{top}(R)$, we denote by S_i the image of \bar{S}_i . Consider $\Xi_i \cong \mathbb{A}^{r_i}$ for some r_i and the morphism $\Xi_i \rightarrow im(\Xi_i)$ can be viewed as the cyclic quotient $\mathbb{A}^{r_i} \rightarrow \mathbb{A}^{r_i} / \frac{1}{m_i}(b_{i_1}, \dots, b_{i_{r_i}})$ for some integers m_i and $b_{i_1}, \dots, b_{i_{r_i}}$. Consider $S_i \subset im(\Xi_i) \cong \mathbb{A}^{r_i} / \frac{1}{m_i}(b_{i_1}, \dots, b_{i_{r_i}}) \subset \mathbb{P}(m_i, b_{i_1}, \dots, b_{i_{r_i}})$ and let \tilde{S}_i be the closure of S_i in $\mathbb{P}(m_i, b_{i_1}, \dots, b_{i_{r_i}})$. Now we have $|\chi_{top}(\tilde{S}_i)| \leq M_{d,k}^{r_i}$ and $|\chi_{top}(\tilde{S}_i - S_i)| \leq M_{d,k}^{r_i-1}$, hence $|\chi_{top}(S_i)| \leq M_{d,k}^{r_i} + M_{d,k}^{r_i-1}$. Moreover, for any $i_1, \dots, i_l \in \{1, \dots, j\}$, $\Xi_{i_1} \cap \dots \cap \Xi_{i_l} \cong \mathbb{A}^{r_{i_1} \dots i_l}$ for some integer $r_{i_1 \dots i_l}$ and the same argument shows that $|\chi_{top}(S_{i_1} \cap \dots \cap S_{i_l})| \leq M_{d,k}^{r_{i_1} \dots i_l} + M_{d,k}^{r_{i_1} \dots i_l-1}$. In particular, let $M = 2 \max_{r < n} \{M_{d,k}^r\}$, then we have $|\chi_{top}(R)| = |\chi_{top}(S_1 \cup \dots \cup S_j)| \leq (2^j - 1)M$ by Lemma 3.12 and $j < 2^n$ as before. The conclusion is that there exists an integer $B_{d,k}^n$ depends only on n, d and k such that $|\chi_{top}(R)| \leq B_{d,k}^n$.

Now we have

$$\begin{aligned} |\chi_{top}(Y')| &= \left| \frac{1}{a_0}(\chi_{top}(\bar{Y}) - \chi_{top}(\bar{R})) + \chi_{top}(R) \right| \\ &\leq |\chi_{top}(\bar{Y})| + |\chi_{top}(\bar{R})| + |\chi_{top}(R)| \leq N_{d,k}^n + A_{d,k}^n + B_{d,k}^n. \end{aligned}$$

Hence $|\chi_{top}(Y)| \leq |\chi_{top}(Y')| + |\chi_{top}(Y'')| \leq M_{d,k}^{n-1} + N_{d,k}^n + A_{d,k}^n + B_{d,k}^n$ can be bounded \square

Proof of Theorem 1.3. One can easily see that $M_{d,k}^1 = N_{d,k}^1 = d$. Hence Proposition 3.1 and Proposition 3.2 implies the theorem. \square

4. THE BOUNDEDNESS OF BETTI NUMBERS

In this section we will bound the variance of b_3 . Thanks to Corollary 2.10, it is equivalence to bound the variance of the topological Euler characteristic, which is much easier to compute. The following statement is a corollary of Theorem 1.3, which could help us to bound the variance of χ_{top} under divisorial contraction to point.

Corollary 4.1. *Assume that X is a cyclic quotient of local complete intersection locus of codimension k in \mathbb{A}^n , and $Y \rightarrow X$ be a weighted blow-up of weight σ . If the σ -weight of the defining equation of X is bounded by a constant d , then $|\chi_{top}(Y) - \chi_{top}(X)| \leq M_{d,k}^n + 1$.*

Proof. Write $\sigma = \frac{1}{m}(a_0, \dots, a_n)$. The exceptional locus E of $Y \rightarrow X$ is contained in $\mathbb{P}^n(a_0, \dots, a_n)$ and is defined by k equations with weight $\leq d$. Hence $|\chi_{top}(E)| \leq M_{d,k}^n$. Now

$$|\chi_{top}(Y) - \chi_{top}(X)| = |\chi_{top}(E) - \chi_{top}(pt)| \leq M_{d,k}^n + 1.$$

\square

Given a divisorial contraction $Y \rightarrow X$, we will show that the difference of χ_{top} can be bound by constant depends only on $dep(X)$ in w -morphism case, and on $dep(Y)$ in general cases. The reason we need the first statement is that inverse of w -morphism occurs in the factorization in Theorem 2.7.

Proposition 4.2. *Let $Y \rightarrow X$ be a divisorial contraction contract a divisor E to a point $P \in X$. Assume the index of P is $m > 1$ and assume $a(E, X) = 1/m$. Then*

$$|\chi_{top}(Y) - \chi_{top}(X)| \leq D_{dep(X)}$$

for some integer $D_{dep(X)}$ depends only on the number $dep(X)$.

Proof. In [Hay1] and [Hay2] Hayakawa classified all extremal divisorial contraction with discrepancy $1/m$ to a higher index point. He proved that such morphism is always a weighted blow-up of a cyclic quotient of local complete intersection locus in \mathbb{A}^4 or \mathbb{A}^5 . We will denote by d the upper bound of the weight of the exceptional locus viewed as a subvariety in the weighted projective space. What we have to do is to show that d can be determined by $dep(X)$ and then $|\chi_{top}(Y) - \chi_{top}(X)| \leq M_{d,k}^n + 1$ for $(n, k) = (4, 1)$ or $(5, 2)$, which is an integer depends only on $dep(X)$. We consider the type of the germs (X, P) .

cA/m . By [Hay1], Theorem 6.4, We have

$$X \cong (xy + f(z, u) = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{m}(\alpha, -\alpha, 1, 0).$$

If we define $\tau\text{-wt}(z) = 1/m$ and $\tau\text{-wt}(u) = 1$, and assume $\tau\text{-wt}(f(z, u)) = k$, then Y is obtained by the weighted blow-up of weight $\frac{1}{m}(a, b, 1, m)$, with $a \equiv \alpha \pmod{m}$ and $a + b = mk$. Furthermore, direct computation shows that Y consists two cyclic quotient singularity, one is of index a and the other one is of index b . We conclude that $dep(X) \geq a + b - 1$ and the exceptional locus can be viewed as a weighted hypersurface in $\mathbb{P}(a, b, 1, m)$ with weight $mk = a + b \leq dep(X) + 1$, hence we take $d = dep(X) + 1$.

$cAx/4$. By [Hay1], Theorem 7.4 and Theorem 7.9, we have $X \cong (\phi = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{4}(1, 3, 1, 2)$ and Y is the weighted blow-up with weight v , where ϕ and v is one of the following.

- (1) $\phi = x^2 + y^2 + f(z, u)$, $v = \frac{1}{4}(2k+1, 2k+3, 1, 2)$ if k is even, and $\frac{1}{4}(2k+3, 2k+1, 1, 2)$ if k is odd, where k is defined by $\tau\text{-wt}(f(z, u)) = (2k+1)/2$, providing $\tau\text{-wt}(z) = 1/4$ and $\tau\text{-wt}(u) = 1/2$. In this case Y consists a cyclic quotient point of index $2k+3$, hence $\text{dep}(X) \geq 2k+3$.
- (2) $\phi = x^2 \pm 2xg(z, u) + y^2 + h(z, u)$, $v = \frac{1}{4}(2k+5, 2k+3, 1, 2)$, where k is defined as above and $\tau\text{-wt}(g) = (2k+1)/4$, $\tau\text{-wt}(h) > (2k+1)/2$. Y consists of a cyclic quotient point of index $2k+5$, hence $\text{dep}(X) \geq 2k+5$.

In the both cases we take $d = 2\text{dep}(X) - 4$.

$cAx/2$. By [Hay1], Theorem 8.4 and Theorem 8.8, we have $X \cong (\phi = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{2}(0, 1, 1, 1)$ and Y is the weighted blow-up with weight v , where ϕ and v is one of the following.

- (1) $\phi = x^2 + y^2 + f(z, u)$, $v = \frac{1}{2}(k, k+1, 1, 1)$ if k is even, and $\frac{1}{2}(k+1, k, 1, 1)$ if k is odd, where k is defined by $\tau\text{-wt}(f(z, u)) = k$, providing $\tau\text{-wt}(z) = 1/2$ and $\tau\text{-wt}(u) = 1/2$.
- (2) $\phi = x^2 \pm 2xg(z, u) + y^2 + h(z, u)$, $v = \frac{1}{2}(k+1, k, 1, 1)$, where k is defined as above and $\tau\text{-wt}(g) = k/2$, $\tau\text{-wt}(h) > k$.

In either case Y consists of a cyclic quotient point of index $k+1$, hence $\text{dep}(X) \geq k+1$ and one can take $d = 2\text{dep}(X) - 2$.

$cD/3$. Use [Hay1], Theorem 9.9, Theorem 9.14, Theorem 9.20 and Theorem 9.25, we may assume $X \cong (\phi = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{3}(2, 1, 1, 0)$ and Y is given by weighted blow-up with weight v , where ϕ and v is given by one of the following.

- (1) $\phi = u^2 + x^3 + yz(y \pm z)$, $v = \frac{1}{3}(2, 4, 1, 3)$, $\frac{1}{3}(2, 1, 4, 3)$.
- (2) $\phi = u^2 + x^3 + yz^2 + xy^4\lambda(y^3) + y^6\mu(y^3)$, $v = \frac{1}{3}(2, 4, 1, 3)$, $\frac{1}{3}(2, 1, 4, 3)$.
- (3) $\phi = u^2 + x^3 + y^3 + xyz^3\alpha(z^3) + xz^4\beta(z^3) + yz^5\gamma(z^3) + z^6\delta(z^3)$, $v = \frac{1}{3}(2, 4, 1, 3)$.
- (4) $\phi = u^2 + x^3 + 3\xi x^2 z^2 + y^3 + xyz^3\alpha(z^3) + xz^7\beta(z^3) + yz^8\gamma(z^3) + z^{12}\delta(z^3)$, $v = \frac{1}{3}(5, 4, 1, 6)$.

The weight only depends on the type, not on the explicit equations. One can take $d = 12$.

$cE/2$. By [Hay1], Theorem 10.11, Theorem 10.17, Theorem 10.22, Theorem 10.28, Theorem 10.33, Theorem 10.41, Theorem 10.47, Theorem 10.54, Theorem 10.61 and Theorem 10.67, we have $X \cong (\phi = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{3}(2, 1, 1, 0)$ and Y is given by weighted blow-up with weight v , where ϕ and v is given by one of the following.

- (1) $\phi = u^2 + x^3 + g(y, z)x + h(y, z)$, $v = \frac{1}{2}(2, 3, 1, 3)$, $\frac{1}{2}(2, 1, 3, 3)$.
- (2) $\phi = u^2 + x^3 + 3\xi x^2 z^2 + g(y, z)x + h(y, z)$, $v = \frac{1}{2}(4, 3, 1, 5)$.
- (3) $\phi = u^2 + x^3 + 3\xi x^2 z^2 + g(y, z)x + h(y, z)$, $v = \frac{1}{2}(2, 1, 3, 3)$.
- (4) $\phi = u^2 \pm 2(\alpha xz + \beta yz^2 + \gamma z^5)u + x^3 + g(y, z)x + h(y, z)$, $v = \frac{1}{2}(4, 3, 1, 7)$.
- (5) $\phi = u^2 + x^3 + g(y, z)x + h(y, z)$, $v = \frac{1}{2}(6, 5, 1, 9)$.

Again in this case one can simply take $d = 18$.

$cD/2$. As in [Hay2], we have the following.

- (1) X is a hyperurface in $\mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{2}(1, 1, 0, 1)$ and Y is a weighted blow-up given by one of the following.

Defining equations	Blowing-up weight	Relations
$u^2 + xyz + x^{2a} + y^{2b} + z^c$	$\frac{1}{2}(1, 1, 2, 3)$ $\frac{1}{2}(1, 3, 2, 3)$ $\frac{1}{2}(3, 1, 2, 3)$ $\frac{1}{2}(1, 1, 4, 3)$	
$u^2 + y^2z + \lambda yx^{2a+1} + g(x, z)$	$\frac{1}{2}(1, l, 4, l+2)$ $\frac{1}{2}(1, 2a+1, 2, 2a+1)$ $\frac{1}{2}(1, b', 2, b')$ $\frac{1}{2}(1, b'-1, 2, b')$ $\frac{1}{2}(1, b'-1, 2, b'+1)$	$l \leq aw(X) + 2$ $2a \leq aw(X)$ $b' \leq aw(X)$
$u^2 \pm 2uh(x, z) + y^2z + \lambda yx^{2a+1} + g(x, z)$	$\frac{1}{2}(1, b', 2, b'+2)$	$b' \leq aw(X)$
$u^2 + y^2z \pm 2yzh(x, z) + \lambda yx^{2a+1} + g(x, z)$	$\frac{1}{2}(1, b'+1, 2, b'+1)$	$b' \leq aw(X)$
$u^2 + g(x, z) + (y - h(x, z))(yz + h(x, z)z + \lambda x^{2a+1})$	$\frac{1}{2}(1, b'+1, 2, b'+1)$	$b' \leq aw(X)$

One can take $d = 6$ in first case. For the other three cases, note that by Lemma 2.4, $aw(X) \leq \Xi(X) \leq 2dep(X)$. Hence we may take $d = 4dep(X) + 4$.

- (2) X is an LCI locus in $\mathbb{A}_{(x,y,z,u,t)}^5 / \frac{1}{2}(1, 1, 0, 1, 1)$ and Y is the weighted blow-up given by one of the following.

Defining equations	Blowing-up weight	Relations
$\begin{cases} u^2 + yt + x^{2a} + z^c \\ t - xz - y^3 \end{cases}$	$\frac{1}{2}(1, 1, 2, 3, 5)$	
$\begin{cases} u^2 + yt + g(x, z) \\ t - yz - \lambda x^{2a+1} \end{cases}$	$\frac{1}{2}(1, 2a-1, 2, 2a+1, 2a+3)$	$2a \leq aw(X)$
$\begin{cases} u^2 + zt + \lambda yx^{2a+1} + q(x) \\ t - y \pm 2uh(x, z) + p(x, z) \end{cases}$	$\frac{1}{2}(1, b', 2, b'+1, 2b'+2)$	$b' \leq aw(X)$
$\begin{cases} u^2 + zt + \lambda yx^{2a-1} + q(x) \\ t - y^2 + p(x, z) \end{cases}$	$\frac{1}{2}(1, b'-1, 2, b'+1, 2b')$	$b' \leq aw(X)$
$\begin{cases} u^2 + yt + g(x, z) \\ t - z(y + 2h(x, z)) + \lambda x^{2a+1} \end{cases}$	$\frac{1}{2}(1, b'-1, 2, b'+1, b'+3)$	$b' \leq aw(X)$

We take $d = 6$ in first case and $d = 4dep(X) + 2$ in the remained four cases.

□

Proposition 4.3. Assume $f : Y \rightarrow X$ is divisorial contraction contract a divisor E to a point $P \in X$. Then there is a number $D'_{dep(Y)}$ depends only on $dep(Y)$ such that

$$|\chi_{top}(Y) - \chi_{top}(X)| \leq D'_{dep(Y)}.$$

Proof. We will discuss case by case.

- (i) P is smooth. By [Kaw1], $Y \rightarrow X$ is a weighted blow-up, so E is a 2 dimensional weighted projective space and then $|\chi_{top}(Y) - \chi_{top}(X)| \leq M_{1,1}^3 + 1$.
- (ii) f is the usual blow-up. As Mori's classification locally X can be viewed as a hypersurface in \mathbb{A}^4 and the exceptional divisor after blowing-up a point will be a degree two surface in \mathbb{P}^3 , hence $|\chi_{top}(Y) - \chi_{top}(X)| \leq M_{2,1}^3 + 1$.

- (iii) P is of index $m > 1$ and $a(X, E) = 1/m$. By Proposition 4.2, $|\chi_{top}(Y) - \chi_{top}(X)| \leq D_{dep(X)}$. Now Proposition 2.1 said that $dep(Y) \geq dep(X) - 1$, hence

$$|\chi_{top}(Y) - \chi_{top}(X)| \leq D_{dep(X)} \leq D_{dep(Y)+1}.$$

- (iv) f is of ordinary type as in [Kaw2]. By [Kaw2], Theorem 1.2, either the divisorial contraction belong to the previous cases, or one of the cases below. We will denote m to be index of P , a be the discrepancy $a(X, E)$ and $J = \{(r_1, 1), (r_2, 1)\}$, $\{(r, 1), (r+2, 1)\}$ or $\{(r, 1), (r+4, 1)\}$ be the non-Gorenstein data of Y defined in [Kaw2]. As the following table one can see that there is a upper bound of the weight of the exceptional divisor which can be written in terms of $dep(Y)$. Note that we have both $r_1 + r_2$ and $2r + 2 \leq \Xi(Y) \leq 2dep(Y)$.

Defining equations	Blowing-up weight	Upper bound
$(xy + g(z^m, u) = 0)$ $\subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{m}(1, -1, b, 0)$	$(r_1/m, r_2/m, a, 1)$	$2dep(Y)$
$(x^2 + xq(z, u) + y^2u + \lambda yz^2 + \mu z^3 + p(y, z, u) = 0) \subset \mathbb{A}_{(x,y,z,u)}^4$	$(r+1, r, a, 1)$	$2dep(Y)$
$(x^2 + xzq(z^2, u) + y^2u + \lambda yz^{2\alpha-1} + p(y, z, u) = 0) \subset \mathbb{A}_{(x,y,z,u)}^4$	$((r+2)/2, r/2, a, 1)$	$dep(Y)$
$\left(\begin{array}{l} yu + z^{(r+1)/a} + q(z, u)u + t = 0 \\ x^2 + xt + p(y, z, u) = 0 \end{array} \right)$ $\subset \mathbb{A}_{(x,y,z,u,t)}^5$	$(r+1, r, a, 1, r+1)$	$2dep(Y)$
$\left(\begin{array}{l} yu + z^{(r+2)/a} + q(z^2, u)zu + t = 0 \\ x^2 + xt + p(z^2, u) = 0 \end{array} \right)$ $\subset \mathbb{A}_{(x,y,z,u,t)}^5 / \frac{1}{2}(1, 1, 1, 0, 1)$	$((r+2)/2, r/2, a, 1, (r+2)/2)$	$dep(Y) + 1$

- (v) f is of exceptional type as in [Kaw2] and $a(X, E) = 1$. In this case we use the results in [Hay3], [Hay4] and [Hay5]. We have several cases here. However the conclusion is, one can take the upper bound to be $\max\{4dep(Y) - 6, 2dep(Y) + 1, 30\}$.

- (a) P is of type cD . By [Hay4] Theorem 2.1-2.5, we have X is a LCI locus in \mathbb{A}^4 or \mathbb{A}^5 and Y is obtained by weighted blow-up with some fixed weight. All the possibility are listed below.

Defining equations	Blowing-up weight	Relation	Upper bound
$x^2 + y^2u + \lambda yz^l + g(z, u)$	$(r, r-1, 1, 2)$	$r+3 \leq \Xi(Y)$	$4dep(Y) - 6$
$x^2 + y^2u + 2yuh(z, u) + \lambda yz^l + g(z, u)$	$(r, r, 1, 1)$	$2r \leq \Xi(Y)$	$2dep(Y)$
$\left\{ \begin{array}{l} x^2 + ut + \lambda yz^l + g(z, u) \\ y^2 + 2xh(z, u) + g'(z, u) - t \end{array} \right.$	$(r+1, r, 1, 1, 2r+1)$	$2r+1 \leq \Xi(Y)$	$2dep(Y) + 1$
$x^2 + 2xh(z, u) + y^2u + \lambda yz^l + g(z, u)$	$(r+1, r, 1, 1)$	$2r+1 \leq \Xi(Y)$	$2dep(Y) + 1$
$\left\{ \begin{array}{l} x^2 + yt + g(z, u) \\ yu + \lambda z^l + 2uh(z, u) - t \end{array} \right.$	$(r, r-1, 1, 1, r+1)$	$2r \leq \Xi(Y)$	$2dep(Y)$

- (b) P is of type cE . By [Hay5] Theorem 1.1, we have X is a LCI locus in \mathbb{A}^4 or \mathbb{A}^5 , and Y is a weighted blow-up as in the following table.

Defining equations	Blowing-up weight
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (2, 2, 1, 1)$
$x^2 + 2xq(y, z) + y^3 + g(z, u)y + h(z, u)$	$w = (3, 2, 1, 1)$
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (3, 2, 2, 1)$
$x^2 + y^3 + 3\xi y^2 u^2 + g(z, u)y + h(z, u)$	$w = (4, 3, 2, 1)$
$x^2 \pm 2x(\alpha y u + \beta z^2 + \gamma z u^2 + \delta u^4) + y^3 + g(z, u)y + h(z, u)$	$w = (5, 3, 2, 1)$
$x^2 + y^3 + 3(\lambda z u + \mu u^3)y^2 + g(z, u)y + h(z, u)$	$w = (5, 4, 2, 1)$
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (6, 4, 3, 1)$
$x^2 + y^3 + 3(\lambda z u + \mu u^4)y^2 + g(z, u)y + h(z, u)$	$w = (7, 5, 3, 1)$
$x^2 \pm 2x u(\alpha y u + \beta z^2 + \gamma z u^3 + \delta u^6) + y^3 + g(z, u)y + h(z, u)$	$w = (8, 5, 3, 1)$
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (9, 6, 4, 1)$
$x^2 + y^3 + 3(\lambda z u^2 + \mu u^6)y^2 + g(z, u)y + h(z, u)$	$w = (10, 7, 4, 1)$
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (12, 8, 5, 1)$
$x^2 + y^3 + g(z, u)y + h(z, u)$	$w = (15, 10, 6, 1)$
$\begin{cases} x^2 + y^3 + l(z, u)t + g(z, u)y + h(z, u) \\ 2l'(z, u)x + g'(z, u)y + h'(z, u) - t \end{cases}$	$w = (3, 2, 1, 1, 5)$
$x^2 + 2(l'(z, u)y + g'(z, u))x + y^3 + g(z, u)y + h(z, u)$	$w = (4, 2, 1, 1)$
$\begin{cases} x^2 + y^3 + q(z, u)t + g(z, u)y + h(z, u) \\ g'(z, u)y + h'(z, u) - t \end{cases}$	$w = (3, 2, 1, 1, 4)$
$x^2 + y^3 - 3q(z, u)y^2 + g(z, u)y + h(z, u)$	$w = (3, 3, 1, 1)$
$\begin{cases} x^2 + yt + g(z, u)y + h(z, u) \\ y^2 + 3(\lambda z u + \mu u^3)y + g'(z, u) - t \end{cases}$	$w = (5, 3, 2, 1, 7)$

The weight of exceptional locus depends only on the type, one may take $d = 30$ as an upper bound.

- (c) P is of type $cD/2$. By [Hay3] Theorem 1.1, X is an LCI locus in

$$\mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{2}(1, 1, 1, 0) \quad \text{or} \quad \mathbb{A}_{(x,y,z,u,t)}^5 / \frac{1}{2}(1, 1, 1, 0, 0)$$

and Y is obtained by weighted blow-up as the following table.

Defining equations	Blowing-up weight	Relation	Upper bound
$x^2 + y^2 u + s(z, u) y u z + r(z) y + p(z, u)$	$(2l, 2l, 1, 1)$	$8l \leq \Xi(Y)$	$\text{dep}(Y)$
$x^2 + y z u + y^4 + z^{2b} + u^c$	$(2, 2, 1, 1)$		4
$\begin{cases} x^2 + ut + r(z)y + p(z, u) \\ y^2 + s(z, u)zx + q(z, u) - t \end{cases}$	$(l + 1, l, 1, 1, 2l + 1)$	$4l + 2 \leq \Xi(Y)$	$\text{dep}(Y) + 1$
$\begin{cases} x^2 + yt + p(z, u) \\ yz + u^2 - t \end{cases}$	$(2l + 2, 2l, 1, 1, 2l + 2)$	$8l + 4 \leq \Xi(Y)$	$\text{dep}(Y)$

- (d) P is of type $cE/2$. By [Hay3] Theorem 1.2, X is defined by

$$(u^2 + x^3 + 3\nu x^2 z^2 + p(x, y, z) = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{2}(0, 1, 1, 1)$$

and Y is obtained by weighted blow-up with weight $(3, 2, 1, 4)$. The weight of exceptional locus is 6.

- (vi) f is of exceptional type as in [Kaw2] and $a(X, E) = 2$. We use the results in [Kaw3]. We have that X is isomorphic to

$$(x^2 + ut + p(y, z, u) = y^2 + q(x, z, u) + t = 0) \subset \mathbb{A}_{(x,y,z,u,t)}^5 / \frac{1}{2}(1, 1, 1, 0, 0)$$

and Y is obtained by weighted blow-up of weight $((r+1)/2, (r-1)/2, 2, 1, r)$ and $2r \leq \text{dep}(Y)$. The weight of exceptional locus can be bounded by $\text{dep}(Y)$.

- (vii) f is of exceptional type, P is Gorenstein and discrepancy is greater than one. We use [Yam], Theorem 2.2-2.10. X is a LCI locus in \mathbb{A}^4 or \mathbb{A}^5 , Y is weighted blow-up and all possible cases are listed below.

Defining equations	Blowing-up weight	Relation	Upper bound
$\begin{cases} x^2 + \lambda yz^k + ut + p(z, u) \\ y^2 + 2xq(z, u) + r(z, u) + t \end{cases}$	$(\frac{r+1}{2}, \frac{r-1}{2}, 4, 1, r)$	$r \leq \Xi(Y)$	$2\text{dep}(Y)$
$\begin{cases} x^2 + \lambda yz^k + ut + p(z, u) \\ y^2 + 2xq(z, u) + r(z, u) + t \end{cases}$	$(\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$	$r \leq \Xi(Y)$	$2\text{dep}(Y)$
$\begin{aligned} &x^2 + y^2u + 2yup(z, u) \\ &+ \lambda yz^k + q(z, u) \end{aligned}$	$(r, r, 2, 1)$	$2r \leq \Xi(Y)$	$2\text{dep}(Y)$
$\begin{aligned} &x^2 + y^2u + 2yup(z, u) \\ &+ \lambda yz^k + q(z, u) \end{aligned}$	$(3, 3, 1, 2)$		6
$\begin{aligned} &x^2 + (y - p(z, u))^3 \\ &+ yg(z, u) + h(z, u) \end{aligned}$	$(3, 3, 2, 1)$		6
$\begin{aligned} &x^2 + y^2 + 2cxy + 2xp(z, u) \\ &+ 2cyq(z, u) + z^3 + g(z, u) \end{aligned}$	$(4, 3, 2, 1)$		6
$\begin{aligned} &x^2 + y^2u + 2yup(z, u) \\ &+ \lambda yz^k + q(z, u) \end{aligned}$	$(3, 4, 2, 1)$		6
$\begin{cases} x^2 + xt + p(z, u) \\ y^2 + q(z, u) + t \end{cases}$	$(5, 3, 2, 2, 7)$		10
$\begin{aligned} &x^2 + y^3 + \lambda y^2u^2 \\ &+ yg(z, u) + h(z, u) \end{aligned}$	$(7, 5, 3, 2)$		14

□

The following lemma treat the blow-up LCI curve case.

Lemma 4.4. *Assume $f : Y \rightarrow X$ is blow up LCI curve C on X , then*

$$\chi_{\text{top}}(Y) - \chi_{\text{top}}(X) = \chi_{\text{top}}(C).$$

Proof. At first one show that over any point of C , the fiber is a \mathbb{P}^1 . To see that, assume C is defined by the ideal I and locally I is generated by the α and β . Then Y is isomorphic to $\text{Proj} \bigoplus_{n \geq 0} I^n$ and the natural map $\mathcal{O}_X[x, y] \rightarrow \bigoplus_{n \geq 0} I^n$ defined by $x \mapsto \alpha$, $y \mapsto \beta$ gives an inclusion $Y \hookrightarrow X \times \mathbb{P}^1$. Hence all fiber along C is a \mathbb{P}^1 . Now there exists a open set $U \subset C$ such that $f^{-1}U \cong U \times \mathbb{P}^1$ since geometric ruled surface are ruled, hence if one denote E to be the exceptional divisor of f , then

$$\chi_{\text{top}}(E) = \chi_{\text{top}}(f^{-1}U) + \chi_{\text{top}}(f^{-1}(C - U)) = 2\chi_{\text{top}}(U) + 2\chi_{\text{top}}(C - U) = 2\chi_{\text{top}}(C)$$

and then

$$\chi_{\text{top}}(Y) - \chi_{\text{top}}(X) = \chi_{\text{top}}(E) - \chi_{\text{top}}(C) = \chi_{\text{top}}(C).$$

□

Now let X be a smooth threefold and consider the process of minimal model program $X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_m = X_{\min}$. We will use above results to estimate the third Betti number of X_i .

Proposition 4.5. *Let $X \rightarrow W$ be a divisorial contraction and $X \dashrightarrow X'$ be flip or flop. Then there is a constant $\Phi_{\text{dep}(X)}$ depends only on $\text{dep}(X)$ such that $b_3(W) \leq \Phi_{\text{dep}(X)} + b_3(X)$ and $b_3(X') \leq \Phi_{\text{dep}(X)} + b_3(X)$*

Proof. Assume $X \rightarrow W$ is a divisorial contraction to point, then by Corollary 2.10 and Proposition 4.3 we have $|b_3(X) - b_3(W)| = |\chi_{\text{top}}(X) - \chi_{\text{top}}(W) - 2| \leq D'_{\text{dep}(X)} + 2$, hence

$$b_3(W) \leq D'_{\text{dep}(X)} + 2 + b_3(X).$$

If $X \rightarrow W$ is blow-up LCI curve on W , then using Corollary 2.10 and Lemma 4.4 one has $b_3(W) - b_3(X) = \chi_{\text{top}}(X) - \chi_{\text{top}}(W) - 2 = \chi_{\text{top}}(C) - 2 \leq 0$, hence

$$b_3(W) \leq b_3(X).$$

If $X \dashrightarrow X'$ is a flop, then $b_3(X) = b_3(X')$. So we only have to consider the cases when $X \rightarrow W$ is a divisorial contraction to curve which is not blow-up LCI curve and $X \dashrightarrow X'$ is a flip.

Note that when X is Gorenstein, there is no flipping contraction and every divisorial contraction to curve is blowing-up LCI curve, as mentioned in Remark 2.3. Hence we may induction on the number $\text{dep}(X)$. By Theorem 2.7 we have the diagram

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ \downarrow f & & \downarrow f' \\ X & & X' \\ & \searrow g & \swarrow g' \\ & W & \end{array}$$

At first note that by Proposition 2.9 and Proposition 4.2,

$$|b_3(Y) - b_3(X)| = |\chi_{\text{top}}(Y) - \chi_{\text{top}}(X)| \leq D_{\text{dep}(X)},$$

hence $b_3(Y) \leq D_{\text{dep}(X)} + b_3(X)$. On the other hand, one write

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \dots \dashrightarrow Y_l = Y',$$

by Remark 2.8 we have $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i > 0$, hence $\text{dep}(Y_{i+1}) < \text{dep}(Y_i)$ for all $i > 0$ and $\text{dep}(Y_0) < \text{dep}(X)$. Thus $l \leq \text{dep}(X)$. By induction hypothesis we have $b_3(Y_{i+1}) < \Phi_{\text{dep}(Y_i)} + b_3(Y_i)$. Now define $\Psi_{\text{dep}(X)}^0 = D_{\text{dep}(X)}$ and $\Psi_{\text{dep}(X)}^n = \Phi_{\text{dep}(X)-1} + \Psi_{\text{dep}(X)}^{n-1}$, then we have

$$b_3(Y_0) = b_3(Y) \leq \Psi_{\text{dep}(X)}^0 + b_3(X)$$

and hence

$$b_3(Y_{i+1}) \leq \Phi_{\text{dep}(Y_i)} + b_3(Y_i) \leq \Phi_{\text{dep}(X)-1} + b_3(Y_i) = \Psi_{\text{dep}(X)}^{i+1} + b_3(X)$$

by induction on i . We conclude that $b_3(Y') = b_3(Y_l) \leq \Psi_{\text{dep}(X)}^{\text{dep}(X)} + b_3(X)$. Finally

$$b_3(X') \leq \Phi_{\text{dep}(Y')} + b_3(Y') \leq \Phi_{\text{dep}(X)-1} + \Psi_{\text{dep}(X)}^{\text{dep}(X)} + b_3(X)$$

since $\text{dep}(Y') < \text{dep}(X)$. So we finish the case when $X \dashrightarrow X'$ is a flip.

Now assume $X \rightarrow W$ is divisorial contraction to curve, then one has to estimate $b_3(W)$. In this case $g' : X' \rightarrow W$ is divisorial contraction to point, hence one may apply Proposition 4.3 to get $|b_3(X') - b_3(W)| = |\chi_{top}(X') - \chi_{top}(W)| \leq D'_{dep(X')}$ and then

$$b_3(W) \leq D'_{dep(X')} + b_3(X') \leq D'_{dep(X)} + \Phi_{dep(X)-1} + \Psi_{dep(X)}^{dep(X)} + b_3(X).$$

□

Proof of Theorem 1.2. (i) and (ii) are Proposition 2.11. Also as in Remark 2.6 we have $dep(X_i) \leq \rho(X)$ for all i . So Proposition 4.5 implies

$$b_3(X_i) \leq \Phi_{\rho(X)} + b_3(X_{i-1}) \leq i\Phi_{\rho(X)} + b_3(X).$$

Now $i \leq 2\rho(X)$ by [CZ], Lemma 3.1. One conclude that one can take $\bar{\Phi}_{\rho(X)} = 2\rho(X)\Phi_{\rho(X)}$. □

5. EXAMPLES AND APPLICATIONS

Let $Y \rightarrow X$ be a extremal divisorial contraction between terminal threefolds, then as Lemma 2.9 $b_i(Y) - b_i(X)$ are known except for b_3 . In the previous section we have shown that $|b_3(Y) - b_3(X)|$ can be bounded by some constant depend only on the depth of X or Y . The following examples shows that the bound is truly depends on the depth. When the depth being larger, the bound should be larger.

Example 5.1. Assume $P \in X$ is locally isomorphic to the origin in

$$(x^2 + y^2 + z^{4k+2} + u^{2k+1} = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{4}(1, 3, 1, 2).$$

This is a isolated terminal point of type $cAx/4$. Assume k is even and let Y be the weighted blow up of weight $\frac{1}{4}(2k+1, 2k+3, 1, 2)$. Then $Y \rightarrow X$ is a extremal divisorial contraction with discrepancy $1/4$. Let E be the exceptional divisor. We have

$$b_3(X) - b_3(Y) = \chi_{top}(Y) - \chi_{top}(X) - 2 = \chi_{top}(E) - 3.$$

Hence to compute $b_3(X) - b_3(Y)$ is equivalent to compute $\chi_{top}(E)$.

Now in this case

$$E \cong (x^2 + z^{4k+2} + u^{2k+1} = 0) \subset \mathbb{P}(2k+1, 2k+3, 1, 2).$$

On $U_z = \{z = 1\}$ we have $E|_{U_z} \cong (x^2 + u^{2k+1} + 1) \subset \mathbb{A}_{(x,y,u)}^3$. This is a line bundle over a smooth curve $C = (x^2 + u^{2k+1} + 1) \subset \mathbb{A}_{(x,u)}^2$ which is of degree $2k+1$, hence

$$\chi_{top}(E|_{U_z}) = \chi_{top}(C) = -(2k-2)(2k+1) - (2k+1),$$

which tends to $-\infty$ as k tends to ∞ .

On the other hand, one can show that $E|_{\{z=0\}}$ is isomorphic to \mathbb{P}^1 . Hence $\chi_{top}(E)$ tends to $-\infty$ when k tends to ∞ . This shows that $b_3(X) - b_3(Y)$ could be arbitrary negative.

Example 5.2. Assume $P \in X$ is locally isomorphic to the origin in

$$(xy + z^{mk} + u^k = 0) \subset \mathbb{A}_{(x,y,z,u)}^4 / \frac{1}{m}(\alpha, -\alpha, 1, m)$$

with $(\alpha, m) = 1$. This is a isolated terminal point of type cA/m . Let Y be the weighted blow up of weight $\frac{1}{m}(a, b, 1, m)$ with $a \equiv \alpha \pmod{m}$ and $a + b = mk$. Then $Y \rightarrow X$ is a extremal divisorial contraction with discrepancy $1/m$. The exceptional divisor E is isomorphic to

$$(xy + z^{mk} + u^k = 0) \subset \mathbb{P}(a, b, 1, m).$$

On the affine open set $U_y = \{y = 1\}$ we have $E|_{U_y} \cong (x + z^{mk} + u^k = 0) \subset \mathbb{A}^3/\frac{1}{b}(a, 1, m)$, which is isomorphic to $\mathbb{A}^2/\frac{1}{b}(1, m)$. One can compute that $\chi_{top}(E|_{U_y}) = 1$.

Now let

$$E' = E|_{\{y=0\}} \cong (z^{mk} + u^k = 0) \subset \mathbb{P}(a, 1, m).$$

We have $E'|_{\{z=1\}} \cong (u^k + 1 = 0) \subset \mathbb{A}_{(x,u)}^2$, which is k lines. Also $E'|_{\{z=0\}}$ is a point, hence

$$\chi_{top}(E') = k + 1.$$

A conclusion is that $\chi_{top}(E) = k + 2$ can be arbitrary large when k growth to infinity, hence $b_3(X) - b_3(Y)$ could be arbitrary positive.

In the rest part we will prove Theorem 1.4. From now on let X be a projective \mathbb{Q} -factorial terminal threefold over \mathbb{C} . For any singular point $P \in X$, we say that there exists a *feasible resolution* for P if there is a sequence

$$X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 = X$$

so that X_n is smooth over P and $X_i \rightarrow X_{i+1}$ is a extremal divisorial contraction to point with minimal discrepancy.

Theorem 5.3 ([C], Theorem 2). *Given a three-dimensional terminal singularity $P \in X$, there exists a feasible resolution for $P \in X$.*

Corollary 5.4. *Given a projective \mathbb{Q} -factorial terminal threefold X over \mathbb{C} , there is a smooth variety Y such that $Y \rightarrow X$ is a composition of steps of K_Y -minimal model program, and the relatively Picard number $\rho(Y/X)$ depends only on the singularity (that is, the local equation near singular points) of X .*

Corollary 5.5. *Notation as above. we have $b_i(Y) \leq b_i(X) + \Theta_i$, where Θ_i is a constant depends only on singularities of X and $\rho(X)$.*

Proof. We apply Theorem 1.2. When $i = 0, 1, 5, 6$, one take $\Theta_i = 0$. For $i = 2, 4$ we choose Θ_i to be $\rho(Y/X)$. Now assume $i = 3$ and assume $Y \rightarrow X$ factors through

$$Y = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 = X$$

with $X_i \rightarrow X_{i+1}$ is a extremal divisorial contraction to point. By Proposition 4.3 and Corollary 2.10 we have

$$|b_3(X_{i+1}) - b_3(X_i)| \leq |\chi_{top}(X_i) - \chi_{top}(X_{i+1})| + 2 \leq D'_{dep(X_{i+1})} + 2.$$

Now n is equal to $\rho(Y/X)$ and $dep(X_{i+1})$ is bounded by $\rho(Y) = \rho(Y/X) + \rho(X)$ (Remark 2.6). Hence

$$|b_3(Y) - b_3(X)| \leq n(D'_{\rho(Y)} + 2)$$

is a constant depends on singularities of X and $\rho(X)$. □

Proof of Theorem 1.4. Let

$$Y = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 = X$$

be a feasible resolution. By [CT] Lemma 2.16 we have

$$0 \rightarrow IH^i(X_j, \mathbb{Q}) \rightarrow IH^i(X_{j+1}, \mathbb{Q}) \oplus IH^i(P_j, \mathbb{Q}) \rightarrow IH^i(E_j, \mathbb{Q}) \rightarrow 0$$

is exact for $i \geq 1$, here $E_j = exc(X_{j+1} \rightarrow X_j)$ and P_j is the image of E_j . Hence $Ib_i(X_{j+1}) \geq Ib_i(X_j)$ for all j . Thus $Ib_i(X) \leq Ib_i(Y) = b_i(Y) \leq b_i(X) + \Theta_i$ by Corollary 5.5. □

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